

Contextual Value-definiteness and the Kochen-Specker Paradox

Ken Williams

Department of Physics

University of Wisconsin - Parkside

Abstract

Compatibility between the realist tenants of value-definiteness and causality is called into question by several realism impossibility proofs in which their formal elements are shown to conflict. We review how this comes about in the Kochen-Specker and von Neumann proofs and point out a connection between their key assumptions: a constraint on realist causality via additivity in the latter proof, noncontextuality in the former. We conclude that value-definiteness and contextuality are indeed not mutually exclusive.

1 overview

In contrast to Bell's theorem which draws a contradiction between the predictions of quantum mechanics and realism, the theorem of Kochen and Specker (KS), "the second important no-go theory against hidden variable theories", rather calls into question the very logic in realist thinking. The argument is directed against a brand of realism characterized by value-definiteness and noncontextuality, formulated here in section 6.1 as propositions $p(1)$ and $p(2)$, respectively. When these are applied to an elementary QM description of spin-1 particle measurements, a contradiction known as the KS paradox follows.

In the concluding view of this article, proposition $p(2)$ does not follow from any realist principle and is even anti-realist in the undue constraint it places on causality. Moreover, no realist to the writer's knowledge lays claim to it. As this question concerns the views of realists, it might well be put to rest with a few excerpts from the writings of well-known realist thinkers. But that would fall short of our broader aim to explain how the misunderstanding of a realist $p(2)$ could have come about, and thus point the way to a reasoned resolution. To this end, we consider the two principal interpretations of quantum mechanics, orthodox and realist, with special attention to their contrasting physical interpretations of the QM wave-function; our approach to the KS paradox is within this context.

We also take the view that the paradox may be well understood from basic quantum mechanics, and so first set about reminding the reader of the relevant elements that he or she may already know from an elementary study of the subject. In section 2 we motivate the discussion of discrete possessed values by considering the quantum signature Stern-Gerlach experimental data, which, in section 3, is described within the framework of quantum mechanics. There, the operational equations and constraints necessary to the KS paradox first appear, equations (18). We then consider in section 4 features of the orthodox interpretation relevant to the evaluation of the equations: In 4.1 the orthodox identification of the QM wave-function with

individual particle states from which follows in 4.2 its view with regard to individual particle possessed-values. Section 5 deals with the realist interpretation, there too, particularly regarding features relevant to evaluation of equations (18). At the heart of the interpretation is the realist identification of the QM wave function with collections of particles, particle ensembles.

The KS paradox is derived in section 6 from propositions p(1) and p(2) applied to the set (18). The connection to the older von Neumann impossibility proof is established in 6.2 and to the later work of Gleason in 6.3. We reconsider the KS paradox in section 7 by close examination of premise p(2). In the end we find little justification for proposition p(2) as a realist tenet, and indeed from the words of realist thinkers, evidence to the contrary. We conclude with an attempt to explain the natural simplicity of contextuality in realist thinking.

2 Stern-Gerlach Data

From the data of Stern-Gerlach (SG) measurements comes a persuasive case that only certain discrete values of intrinsic angular momentum, spin, are observable. There, a stream of unprepared electrons enter and exit a small region of intense inhomogeneous magnetic field strength where it divides in two, half the electrons deflecting up, the other half down by the same set magnitude [1] proportional to a given electron's spin projection along the SG apparatus symmetry axis. Thus measured, each spin projection is found to have one of the two values, $\pm\frac{1}{2}$, for which reason the electron is called a spin- $\frac{1}{2}$ particle. A second measurement along the same SG axis taken on either of the two sub-ensembles invariably confirm the previous result: electrons previously deflected up (down) due to positive (negative) spin of magnitude $\frac{1}{2}$ along the axis, are again deflected up (down)

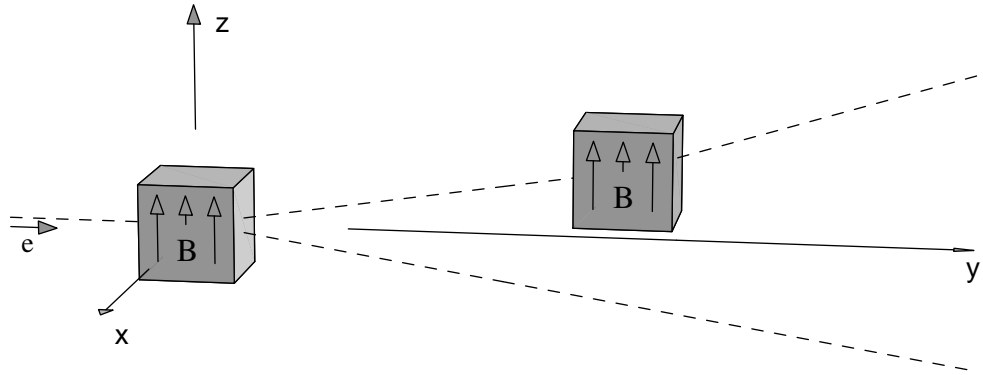


figure 1

where $\hat{\mathbf{B}}$ in figure 1 gives the averaged SG B-field direction. As the pre-measured spin orientations are assumed random, isotropically distributed and uncorrelated to the orientation of the measuring apparatus, this phenomena is unexpected. One would from classical considerations expect the deflection magnitudes to vary continuously with the relative spin-to-apparatus angle

$$d \propto s_z = s \cos \phi \quad (1)$$

having maximum and minimum values for $\phi = 0 \& \frac{\pi}{2}$, respectively

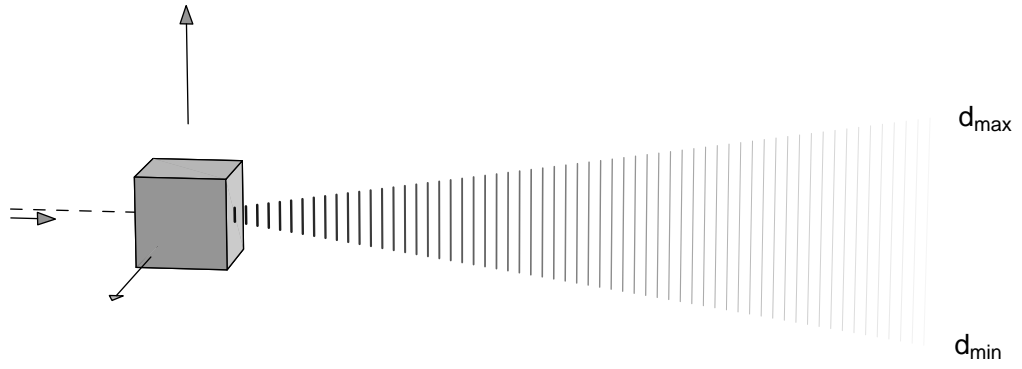


figure 2

One finds however that when the apparatus itself is re-oriented (to test against apparatus-to-ensemble correlation), the same result obtains

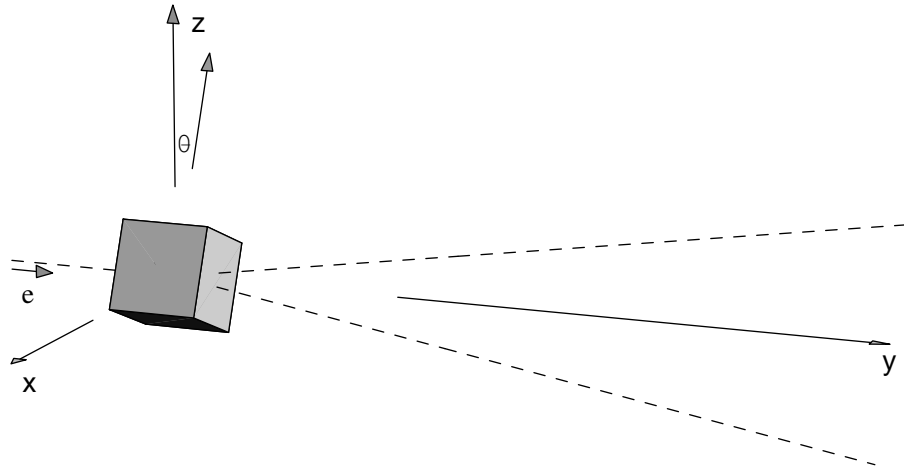


figure 3

half electrons deflecting up again by the same magnitude as before, half down, only now along the re-oriented axis. Moreover, when this second measurement is taken along an axis perpendicular to the first, again, half are deflected up, half down, by the same magnitude. Indeed, one finds that no matter the SG orientation or the order in which one appears in a sequence of such measurements, the results are always $+\frac{1}{2}$ or $-\frac{1}{2}$ only. The electron's spin projection is therefore said to be quantized (i.e., given in discrete, set amounts) along the measurement axis, called then the axis of quantization, in the figure below the SG symmetry axis specified by θ and θ' , SG orientations relative to the laboratory z-axis.

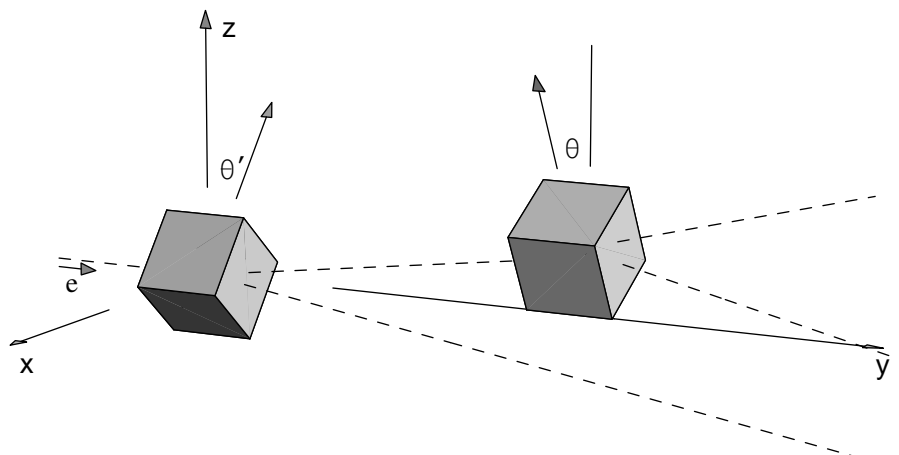
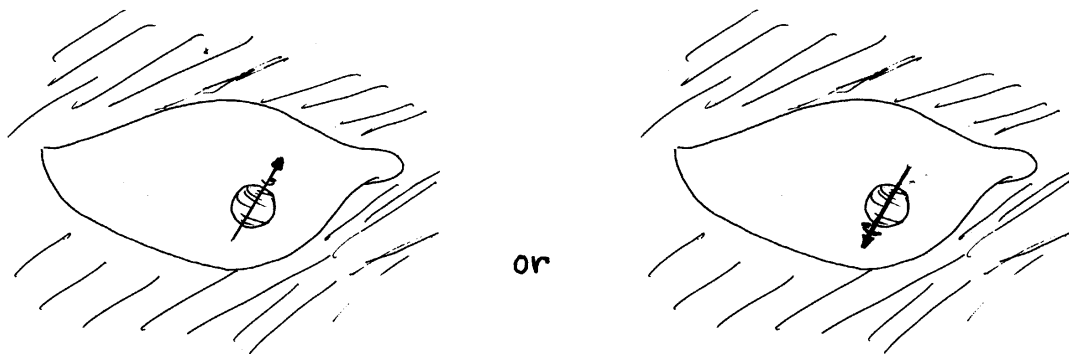


figure 4

The phenomena poses a difficulty to ordinary intuition. Imagine e.g. that whenever you looked up at the moon its polar tilt (as measured from the plane of sight) varied in 180° increments only



no matter the orientation of your head (as measured from the local earth plane).

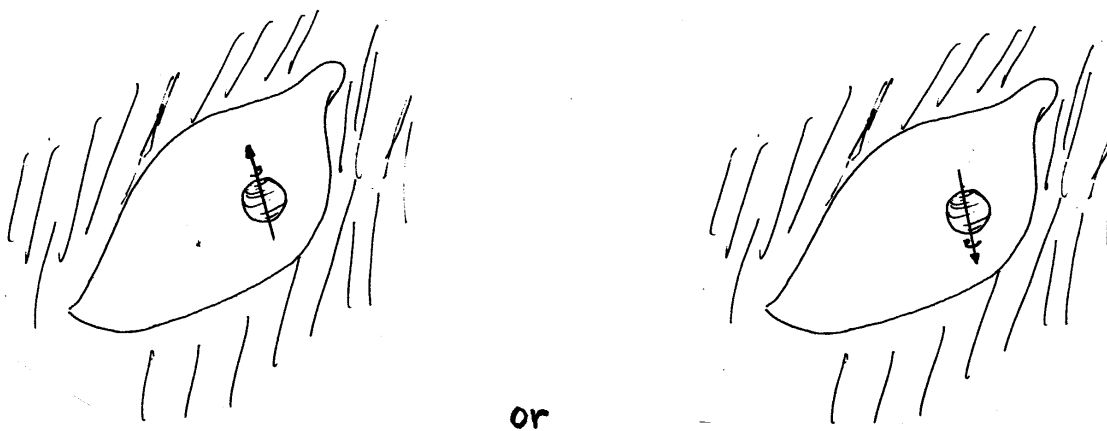


figure 5

The moon's angular momentum would then be said to be quantized.... For the electron this and other unusual microscopic phenomena are well described by the Physics of small-scales, quantum mechanics.

3 quantum mechanical description

Relative frequencies for each of the spin-up, spin-down classes of electron data described above may be obtained from the formalism of quantum mechanics. Experimentally, the frequencies are statistical probabilities over ensembles, collections of identical elements, strictly valid in the large ensemble limit. As regards individual measurements, however, the theory makes no prediction ¹. The up (down) deflection probability at apparatus θ for a sub-ensemble of electrons previously deflected up at apparatus θ' is given by $\cos^2 \frac{\phi}{2}$ ($\sin^2 \frac{\phi}{2}$), $\phi \equiv \theta - \theta'$, assuming for simplicity SG rotation with respect to the laboratory y-axis.

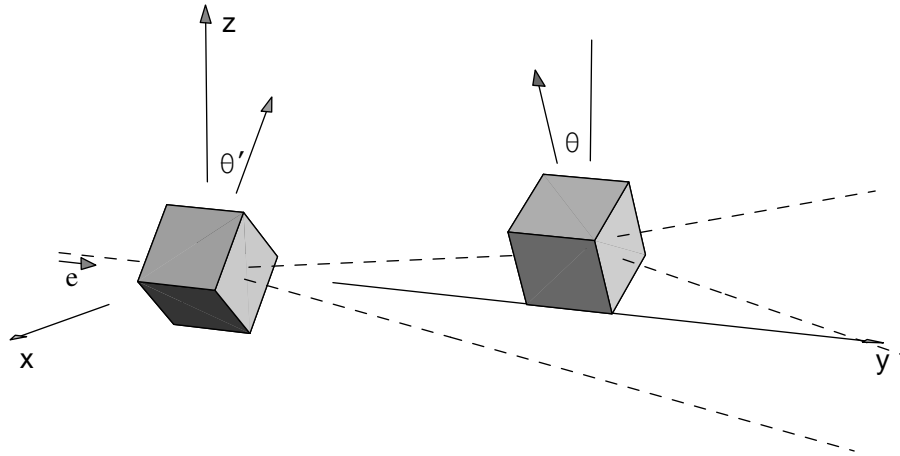


figure 6

These probabilities are expectation values of projector operators $P_{\theta\pm}$ in a complex Hilbert space of state vectors $|\Psi\rangle$ whose calculation proceeds as follows:

$$\langle P_{\theta\pm} \rangle =_{\theta'} \langle + | P_{\theta\pm} | + \rangle_{\theta'} = \begin{cases} \cos^2 \frac{\phi}{2} \\ \sin^2 \frac{\phi}{2} \end{cases} \quad (2)$$

¹See e.g. ref. [2, p. 31] and ref. [3, p. 2]

where

$$|+\rangle_\theta = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} |-\rangle \quad (3)$$

$$P_{\theta\pm} = |\pm\rangle_\theta \langle \pm|_\theta \quad (4)$$

with relations

$$\begin{aligned} \langle \pm | \pm \rangle &= 1 \\ \langle \pm | \mp \rangle &= 0. \end{aligned} \quad (5)$$

For deflected-down prepared subensemble predictions, vector $|+\rangle_{\theta'}$ in (2) is replaced by $|-\rangle_{\theta'} = \cos \frac{\theta'}{2} |-\rangle - \sin \frac{\theta'}{2} |+\rangle$.

Spin-up/spin-down vectors, $|\pm\rangle$, are so-called by analogy with the relation between ordinary classical spin vectors, \mathbf{s}_\pm , and what would be their SG deflections. However in the two-dimensional Hilbert space H2 which they span, the directions of $|\pm\rangle$ are not antiparallel, but mutually perpendicular, corresponding to dual, mutually exclusive experimental outcomes.

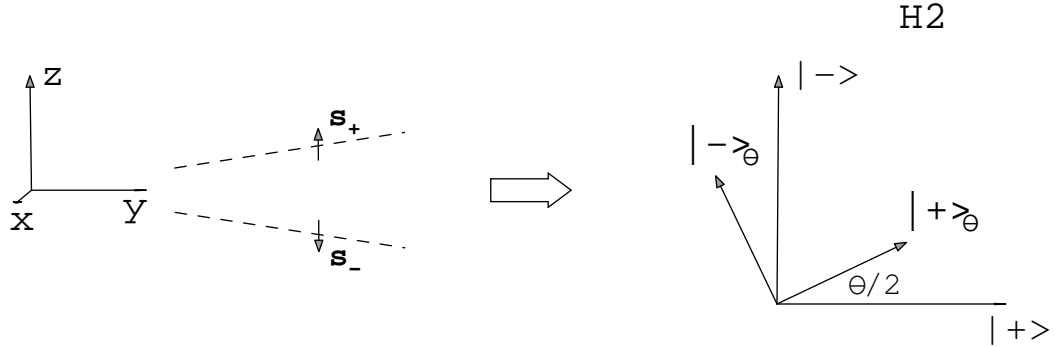


figure 7

Given projectors $P_{\theta\pm}$, both vectors are completely specified by relations

$$\begin{aligned} P_{\theta\pm} |\pm\rangle_\theta &= (1) \cdot |\pm\rangle_\theta = |\pm\rangle_\theta \\ P_{\theta\pm} |\mp\rangle_\theta &= (0) \cdot |\mp\rangle_\theta = 0 \end{aligned} \quad (6)$$

called eigenvalue (proper- value) relations, where (6) states that $|+\rangle_\theta$ is an eigenvector of both projector operators $P_{\theta+}$ and $P_{\theta-}$ having eigenvalues 1 and 0, respectively; these are sometimes called "yes" and "no" eigenvalues, and projectors, accordingly, yes-no operators. In consequence, $|\pm\rangle_\theta$ are also eigenvectors of the quantum mechanical spin operator, $S_\theta = \frac{1}{2}(P_{\theta+} - P_{\theta-})$, having eigenvalues $\pm\frac{1}{2}$

$$S_\theta|\pm\rangle_\theta = \pm\frac{1}{2}|\pm\rangle_\theta \quad (7)$$

which gives immediately that the vectors $|\pm\rangle_{\theta'}$ with $\theta' \neq \theta$ are generally not spin S_θ eigenvectors.

Vectors and operators may also be expressed in 2-dimentional matrix form

$$|+\rangle_\theta = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} \quad |-\rangle_\theta = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} \quad (8)$$

$$P_{\theta+} = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \frac{1}{2} \sin \theta \\ \frac{1}{2} \sin \theta & \sin^2 \frac{\theta}{2} \end{pmatrix} \quad P_{\theta-} = \begin{pmatrix} \sin^2 \frac{\theta}{2} & -\frac{1}{2} \sin \theta \\ -\frac{1}{2} \sin \theta & \cos^2 \frac{\theta}{2} \end{pmatrix} \quad (9)$$

whereby relations (2), (5), and (6) above are of course preserved. In addition, for each quantization θ we have the projector relations

$$P_{\theta+} + P_{\theta-} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \mathbf{1} \quad (10)$$

$$P_{\theta\pm}^2 = P_{\theta\pm} \quad (11)$$

known respectively as completion and projector conditions. Their symbolic evaluation over the ensemble yields the expectation value equation

$$\langle P_{\theta+} \rangle_\theta + \langle P_{\theta-} \rangle_\theta = 1 \quad (12)$$

with constraint

$$\langle P_{\theta\pm} \rangle_\theta = 1 \text{ or } 0. \quad (13)$$

It is sometimes of interest to compare sets of eigenvector pairs for selected quantizations. From (8) we tabulate for orientations $\{\theta\} = \{0, \pi/2, \pi, 3\pi/2\}$.

θ	0	$\pi/2$	π	$3\pi/2$
$ +\rangle_{\theta}$	(1,0)	(1,1)/ $\sqrt{2}$	(0,1)	(-1,1)/ $\sqrt{2}$
$ -\rangle_{\theta}$	(0,1)	(-1,1)/ $\sqrt{2}$	(-1,0)	(-1,-1)/ $\sqrt{2}$

table 1

Notice that eigenstates are not necessarily exclusive to single orientations: $|-\rangle_{\pi/2} = |+\rangle_{3\pi/2}$, so that e.g. from (4), $P_{\pi/2-} = P_{3\pi/2+}$. As the ambiguity is significant to the coming discussion, for emphasis we now make a notational change. We specify a Hilbert space eigenstate now by the spatial direction of its classical counterpart, one related to the other as illustrated in figure (7). Then, $|\pm\rangle_{\theta'} \sim |\mathbf{s}'_{\pm}\rangle$. P'_i thus projects out the QM state with classical counterpart \mathbf{s}'_i measured by an SG apparatus with orientation θ' , which, interchangeably with the corresponding average SG field direction $\hat{\mathbf{B}}'$, we shall call the measurement's *context*. The two notations are related by

$$\begin{aligned}
\hat{\mathbf{B}}' \cdot \hat{\mathbf{k}} &= \cos(\theta') \\
\hat{\mathbf{s}}'_{1,2} &\equiv \hat{\mathbf{s}}'_{+,-} = +, -\hat{\mathbf{B}}' \\
P'_{1,2} &= P_{\theta'+,-}
\end{aligned} \tag{14}$$

We therefore have from table 1

$$P_2^{(\pi/2)} = P_1^{(3\pi/2)} \tag{15}$$

consequent upon $\mathbf{s}_2^{(\pi/2)} = \mathbf{s}_1^{(3\pi/2)} (= \hat{\mathbf{i}})$: The spin-down SG outcome state for SG orientation $\theta = \pi/2$ is the same as the spin-up outcome state for SG orientation $\theta = 3\pi/2$. In the new notation table 1 becomes

B	k	-i	-k	i
$ s_1 >$	(1,0)	(1,1)/ $\sqrt{2}$	(0,1)	(-1,1)/ $\sqrt{2}$
$ s_2 >$	(0,1)	(-1,1)/ $\sqrt{2}$	(-1,0)	(-1,-1)/ $\sqrt{2}$

table 2

For the purposes of the KS analysis here we will need to consider the structure of measurement outcomes on spin-1 particles, measurements also made by the SG apparatus. The observed mutually exclusive outcomes are: deflection up, deflection down, and no deflection at all.

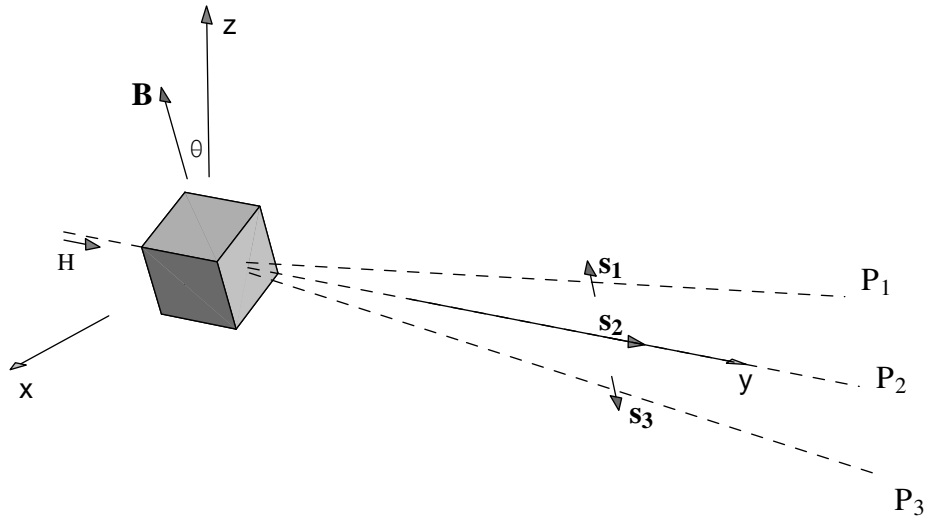


figure 8

In the now corresponding *three* dimensional Hilbert space of states we have operator relations

$$P_1 + P_2 + P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{1} \quad (16)$$

$$P_i^2 = P_i \quad (17)$$

and related expectation-value completion and projection relations

$$\begin{aligned} \langle P_1 \rangle' + \langle P_2 \rangle' + \langle P_3 \rangle' &= 1 \\ \langle P_i \rangle' &= 1 \text{ or } 0 \quad \text{for } \mathbf{B}' = \mathbf{B} \end{aligned} \quad (18)$$

a constrained sum rule (csr) central in the derivation of the KS paradox. Looking ahead, we now tabulate the eigenvector triplets for the four contexts $\{\mathbf{B}\} = \{\hat{\mathbf{k}}, -\hat{\mathbf{i}}, -\hat{\mathbf{k}}, \hat{\mathbf{i}}\}$

B	k	-i	-k	i
$ \mathbf{s}_1\rangle$	(1,0,0)	(1/√2,1,1/√2)	(0,0,1)	(1/√2,-1,1/√2)
$ \mathbf{s}_2\rangle$	(0,1,0)	(1,0,-1)	(0,1,0)	(1,0,1)
$ \mathbf{s}_3\rangle$	(0,0,1)	(1/√2,-1,1/√2)	(1,0,0)	(1/√2,1,1/√2)

table 3

Table 3 and csr (18) taken together lend themselves to various physical interpretations, most of which may be grouped into one of two main categories: orthodox and realist. The KS contention detailed in section 6 is that an expanded table 3 (obtained simply by expanding the number of contexts considered) is not amenable to any self-consistent realist interpretation; this is the KS paradox. For this reason, we begin in the next two sections by considering such features of the two broad interpretations as are relevant to the KS discussion.

4 Orthodox interpretation of SG data and QM description

4.1 Idea of the individual electron state

There are several ways in which the behavior of individual electrons under the influence of SG measurements is similar to that of quantum-mechanical spin eigenfunctions under the action of projector operators. As a prepared electron deflects either up or down along a measuring SG axis, a projector operator will project an arbitrary state Ψ into either a spin-up or a spin-down eigenstate characteristic of the SG orientation $\hat{\mathbf{B}}$ ².

$$P|\Psi\rangle \xrightarrow{\text{collapse}} |\mathbf{s}_1\rangle \text{ or } |\mathbf{s}_2\rangle \quad (19)$$

An electron spin prepared along a given measurement axis then will not be altered upon measurement - a deflected up(down) prepared electron deflects up(down) again - just as projector operators project their own eigenstates onto themselves. On the other hand, if the prepared and measurement axis are not identical, the electron deflects up or down with probability amplitude given by the corresponding state projected eigenfunction coefficient

$$P'_{1,2}|\mathbf{s}_1\rangle = \begin{cases} \cos \frac{\phi}{2} |\mathbf{s}'_1\rangle \\ \sin \frac{\phi}{2} |\mathbf{s}'_2\rangle \end{cases} \quad (20)$$

illustrated

²The actual projection on an arbitrary state is by prescription only; a state is said to "collapse" upon measurement to one of the context's eigenstates, though no entirely satisfactory dynamics for the collapse has ever been worked out [4]. The shortcoming is closely related to the so-called measurement problem in QM.

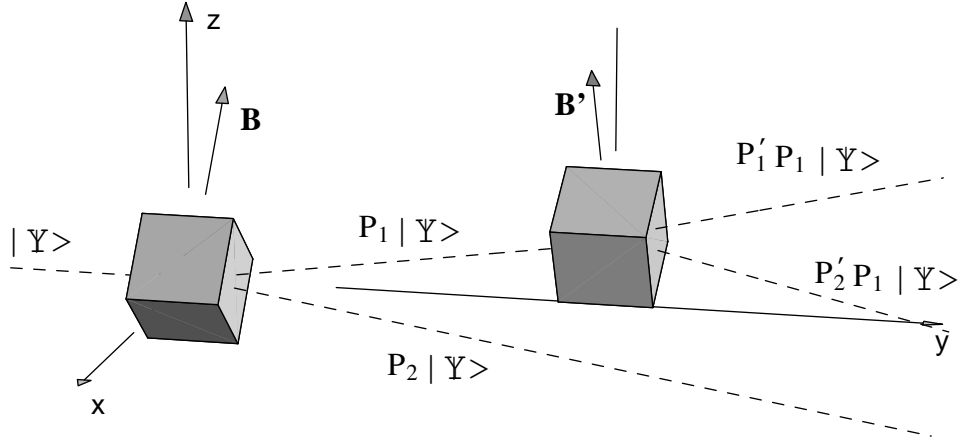


figure 9

And so one might well identify the quantum mechanical spin- $\frac{1}{2}$ wave function $|\Psi\rangle$ with individual electron states, projector operators with the measuring apparatus, and action of the operator on the wave function with the measuring process, just the identifications in fact made in the conventional or orthodox interpretation of QM³. While the assignments are largely unimportant to the practical workings of the formalism, the calculation of expectation values, they offer a picture of the world of individual electrons, a kind of metaphysics.

One difficulty with the interpretation is that spin-vectors are particular to their quantization, having definite projections only along the quantized directions of their measurement axis; in all other directions the projection is not well-defined. The same then in this interpretation for the electron spin itself; the orthodox electron does not possess a spin-projection along a given direction except upon measurement (and prior to subsequent distinct measurements); unlike ordinary classical spin projections, an electron's seems to exist only along one set of directions at a time.

³See e.g. ref. [1, pp. 108 & 214] .

Another difficulty is that "identical" electrons - those of a given ensemble and so represented by the same spin function - generally exhibit different behaviors when subjected to the same conditions: Some electrons of a deflected-up subensemble e.g. will deflect up, others down, when their spin projections are measured along a direction different from the preparation axis. The phenomena is at odds with the principle of the identity of indiscernibles [5] (see also discussion in [6, p. 8], and "sufficient reason" principle in [7, p. 266]).

Finally, the probabilities of QM have meaning to the experimentalist only in reference to ensemble measurements.⁴

4.2 Orthodox interpretation of csr (18)

The value of the expectation $\langle P_i \rangle_\Psi$ answers the question whether a particle in state Ψ has a definite spin in context \mathbf{B} : 1 or 0 for yes, no otherwise. From (18) we see that the answer is "yes" generally when the particle has been prepared within the measurement context of interest, $|\Psi\rangle \rightarrow |\mathbf{s}'_j\rangle$ with $\mathbf{B}' = \mathbf{B}$. But the answer might also be yes for a different preparation context, $\mathbf{B}' \neq \mathbf{B}$, provided the classical companions to Hilbert space vectors $|\mathbf{s}'_j\rangle$ and $|\mathbf{s}_i\rangle$, \mathbf{s}'_j & \mathbf{s}_i , coincide for some i and j. We illustrate such an instance

⁴see ref. [3, p. 2] and [2, p. 36]

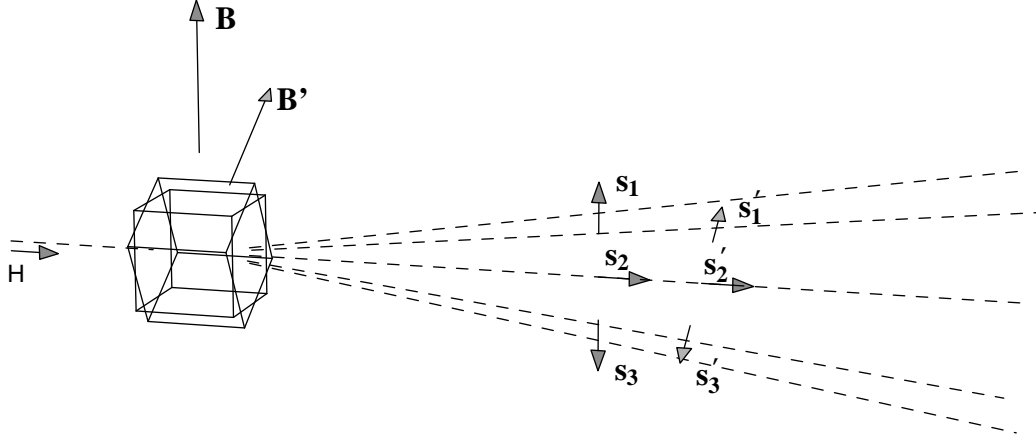


figure 10

In this case the projection measurement with result corresponding to that for classical spin direction $\mathbf{s}_2 = \mathbf{s}'_2$ can be made within either context \mathbf{B} or \mathbf{B}' with equal results, and is therefore said to be context independent (wrt contexts \mathbf{B} and \mathbf{B}'), or noncontextual. A prepared particle $|\mathbf{s}'_j\rangle$ might therefore have a definite spin in all contexts with quantization \mathbf{s}'_j and can be said to *possess* the corresponding set of projector operator eigenvalues. In all other contexts its spin projections are not definite, and the particle could not be said to possess the corresponding projector eigenvalues.

It is easy to see that the number of noncontextuality constraints imposed by this interpretation on a set of sum rules (18) corresponding to a large number of contexts considered simultaneously would be relatively small. Not so in the realist interpretation, as we shall see in the next section. It is this consequence of the realist interpretation that the KS analysis in the section following exploits to draw a self-contradiction.

5 Realist interpretation of SG data and QM description

5.1 Deterministically-held possessed values and the realist extension to csr (18)

In the realist view there is a sense in which an electron has definite spin projections in all directions at all time; its physical characteristics combined with preparation determines *unambiguously* whether an electron will deflect up or down in a given Stern-Gerlach measurement, nothing left to chance. "The good lord does not throw dice", is how Einstein is reported to have put it [7, p. 190]. Then, in the event that otherwise identical electrons have different deflections their preparations, their causal histories, must also have been different.

In this sense it might be said that an electron is in possession of a deflection value, "up" or "down", and hence in possession of its spin projection, prior to measurement. In much the same sense we say that a massive object possess a weight, referring to the weight value $\nu(w)$ a scale will read when the object is placed on it. This does not conflict with the possibility that one scale gives a different reading from another; one may speak of a body's terrestrial weight, its weight on the moon, on mars, etc.. A massive body, like the realist electron, is at all times in possession of an infinite number of measurement values, the set of values that (for the body) weight scales may measure for each of the infinitely many possible measurement situations. These values are deterministically well-defined and so, given the object's physical characteristics and the deterministic factors acting upon it, the causal chain leading up to the measurements, may be identified with that object. By this identification - in this sense - we say that the values are possessed by the object - weights of a body, spin projections of an electron.

A body's weight values might of course be determined from its mass, given the relevant celestial masses and the law of gravity; in a world of weight measurements

however a statement of either is equivalent. Likewise, in absence of a knowledge of some underlying electron spin and the dynamics of microscopic spin-measurements, we may meaningfully identify a set of possible measurement results $\{\nu(s)\}$ with an electron. Fundamental to the realist view is that an object's possessed values are, in principal, verifiable measurement values.

The description is compatible with the realist tenet that

$$\textit{Distinct phenomena supervene upon distinct causes only}^5. \quad (21)$$

on which is based the EPR reality criterion

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality to this physical quantity.

which set the stage for their famous argument in defense of realism [9] and lead eventually to the Bell inequality constraint on the same [10]. In Bell's analysis, the above description of electron possessed values $\{\nu(s)\}$ also appear, depending too upon the manner in which the spin measurements are taken, their contexts. And it is only upon stringent space-time constraints that context dependence is ruled out. We now use the notation $\{\nu(s)\}'$ to denote the set of electron spin projection values measurable in context \mathbf{B} , though possessed by the particle while in context \mathbf{B}' . Similarly for the set of projector (yes-no) values

$$\{\nu(P)\}' \equiv \{\nu'(P_1), \nu'(P_2), \dots \nu'(P_n)\} \quad (22)$$

where $\nu'(P_i)$ is the yes-no value for the i^{th} quantized direction (corresponding projector eigenvalues in descending order) of context \mathbf{B} , possessed by a particle while in context \mathbf{B}' . We identify realist possessed values with QM operators in this notation only for the sake of comparison with QM expectation values with which they sometime coincide; the value $\nu(P)$ simply answers the question whether or not a

⁵cf. opposing accounts of the "realist view" in [6, p. 19] and [8, pp. 12-13]

spin-measured quantum particle would scatter in the same direction as a classical particle of spin \mathbf{s} , and is not to be understood as a formal mapping of the QM projector P . The correlation between expected and experimental values is of primary importance in the realist view, that between expected values and the QM formalism (or any other) of secondary importance; the extent to which eigenvalues of a QM operator are observed values is the same to which the realist possessed value $\nu(\mathbf{O})$ is an eigenvalue of the QM operator \mathbf{O} .

By realist extension of csr (18) to all contexts, then, we have:

$$\begin{aligned}\nu'(P_1) + \nu'(P_2) + \nu'(P_3) &= 1 \\ \nu'(P_j) &= 1 \text{ or } 0 \quad \forall \mathbf{B}'.\end{aligned}\tag{23}$$

Lastly, we may attribute to a hypothetical realist state function, one completely describing the state of an individual particle, the complete set of the particle's possessed values

$$\psi = \psi[\{\{\nu(P)\}'\}] \quad \forall \mathbf{B}, \mathbf{B}'.\tag{24}$$

The realist particle thus represented by ψ has definite spin projections in all H-spatial directions, simultaneously in all contexts.

5.2 Relation between realist ψ and QM Ψ

We symbolically represent the value assignment for an electron-spin measurement

$$\langle \psi | \mathbf{S} | \psi \rangle = \nu(\mathbf{s}) \rightarrow \pm \frac{1}{2}.\tag{25}$$

For sets of realist possessed values of mutually non-commuting contexts, however,

$$[\mathbf{S}_\theta, \mathbf{S}_{\theta'}] \neq 0,\tag{26}$$

QM predicts and observation confirms that a measurement of one set alters the elements of the other; determination of such simultaneous possessed value sets is therefore not possible.

We illustrate a possible difficulty to realist thinking that this resolves by first reverting back briefly to the former notation and consider the QM operational identity

$$\mathbf{S}_{\pi/4} = (\mathbf{S}_0 + \mathbf{S}_{\pi/2})/\sqrt{2} \quad (27)$$

with the above realist value assignments (23) applied to its QM expectation

$$\langle \Psi | \mathbf{S}_{\pi/4} | \Psi \rangle = (\langle \Psi | \mathbf{S}_0 | \Psi \rangle + \langle \Psi | \mathbf{S}_{\pi/2} | \Psi \rangle) / \sqrt{2} \rightarrow \pm \frac{1}{2} = (\pm \frac{1}{2} \pm \frac{1}{2}) / \sqrt{2} \quad (28)$$

a statement manifestly false for all \pm combinations. The mistake results from an essential difference between the eigenvalues $\nu(\mathbf{O})$ and expectation values $\langle \Psi | \mathbf{O} | \Psi \rangle$ of an operator \mathbf{O} . We refer however to the use of relation (27) for its *meaning* which lies in the agreement obtained between its expectation value and the experimental average of measurements made upon many, many electrons for each of the three indicated contexts, i.e. in an agreement with empirical relations between sub-ensemble averages. In practice, then, (27) pertains to three entirely disjoint electron sub-ensembles, each measured within one of the three contexts, $\{0, \pi/4, \pi/2\}$, so that no single electron spin is measured more than once; the meaning of (27) derives from the identification of quantum mechanical state functions Ψ with subensembles [11], and not from a direct association with the possessed values of individual ensemble members described by the realist ψ . With $|\Psi\rangle$ as the QM state function, the expectation value relation

$$\langle \Psi | \mathbf{S}_{\pi/4} | \Psi \rangle = (\langle \Psi | \mathbf{S}_0 | \Psi \rangle + \langle \Psi | \mathbf{S}_{\pi/2} | \Psi \rangle) / \sqrt{2} \quad (29)$$

is found to correspond to the observed relation

$$\bar{\mathbf{S}}_{\pi/4} = (\bar{\mathbf{S}}_0 + \bar{\mathbf{S}}_{\pi/2}) / \sqrt{2} \quad (30)$$

where

$$\bar{\mathbf{S}}_{\theta} \equiv \frac{1}{2}(n_+ - n_-) / N \quad (31)$$

n_{\pm} the number of ensemble electrons observed to deflect up/down in context θ , and $N = n_+ + n_- \rightarrow \infty$, the total number of electrons. Although the QM formalism does not itself attach to its state functions a physical interpretation, in use Ψ is clearly to be associated with sub-ensembles, in line with its realist interpretation ⁶. An expansion of the QM Ψ over hypothetical realist electron states ψ might take the form

$$\begin{aligned} |\Psi\rangle &= (n_+|+\rangle_{\theta} + n_-|-\rangle_{\theta})/N \\ &= (n_1|\mathbf{s}_1\rangle + n_2|\mathbf{s}_2\rangle)/N \end{aligned} \quad (32)$$

with

$$|\mathbf{s}_i\rangle = \sum_{\{\nu(\mathbf{P}_i)=1\}} |\psi[\{\nu(\mathbf{P}')\}''\rangle] \quad (33)$$

normalized to 1. Such an expansion leads immediately from the QM prediction (29) to the observed (30). The terms averaged over, $\{\nu(\mathbf{P}')\}''$, with their contextual dependence, may be understood as the sub-ensemble's \mathbf{B} -context hidden variables. As $\hat{\mathbf{B}}$ context eigenstates $|\mathbf{s}_i\rangle$ will generally not have definite values in other contexts, we have

$$0 \leq \langle \mathbf{s}'_j | \mathbf{P}_i | \mathbf{s}'_j \rangle \leq 1 \quad (34)$$

although for realist possessed values

$$\nu'(\mathbf{P}_i) = 1 \text{ or } 0 \quad \forall \quad \mathbf{B}'. \quad (35)$$

A QM state function $|\Psi\rangle$ of (32) on the other hand has definite values in at most one context and is therefore said in the realist view to give a physically incomplete picture of the individual electron. This difference forms the basis of our reconsideration of the KS paradox in section 7.

⁶For an alternative and unconventional view see [12]. With regard to the special-case arguments put forward there, one might also consider the special case of homogeneous ensembles.

6 Kochen-Specker paradox

6.1 derivation of the paradox

In line with the discussion in section 5 the following might be proposed as a realist proposition

p(1): All real state quantities (spin, mass, etc.) have definite values at all times.

which is known as the principle of value-definiteness. Pairing with this a second proposition

p(2): The value of a real quantity does not depend on how it is measured

called noncontextuality, and applying them together to the csr's (23) draws a contradiction in any Hilbert space of greater than two dimensions. As the Hilbert space dimension for a particle of spin "s" is given by $2s+1$, KS shows that it is not possible to apply both p(1) and p(2) consistently for a particle of spin $s \geq 1$; this is the KS paradox [13, 14].

Let us consider the simultaneous application of the propositions on (23). From p(1) all equations and constraints hold simultaneously at this time

1. $\nu'(P) = 0 \text{ or } 1$

2. for all \mathbf{B}'

and from p(2) the values are independent of context:

1. $\nu'(P) = \nu''(P) \rightarrow \nu(P)$.

In their original work the authors next proceed with a lengthy formal argument, a simplification of which is the following :

The numbers $\nu(P)$ are the eigenvalues of Hilbert space vectors $|\mathbf{s}_j\rangle$. According to the realist csr's (23) it must then be possible to assign to each mutually orthogonal set

of three vectors the values 0, 0, and 1. As the Hilbert space of QM state vectors, $H3$, is complex, a violation of this rule in a Real space of the same dimension will suffice for the negative proof of interest ($R3$ being a subset of $H3$). Note however that the corresponding real vectors in the mapping $QM : R3 \rightarrow H3 \supseteq R3$ (analogous to the 2-dimensional case pictured in figure 7) do not coincide.

To begin, KS show in the following way that two vectors separated by an angle

$$\phi \leq \cos^{-1}(\sqrt{8/3}) \quad (36)$$

must have the same eigenvalue assignment, 0 or 1. Consider the three sets of mutually orthogonal triplets

$$\{|\mathbf{s}_1^{(1)}\rangle, |\mathbf{s}_2^{(1)}\rangle, |\mathbf{s}_3^{(1)}\rangle\}, \quad \{|\mathbf{s}_1^{(2)}\rangle, |\mathbf{s}_2^{(2)}\rangle, |\mathbf{s}_3^{(2)}\rangle\}, \quad \{|\mathbf{s}_1^{(3)}\rangle, |\mathbf{s}_2^{(3)}\rangle, |\mathbf{s}_3^{(3)}\rangle\} \quad (37)$$

in $R3$ together with a tenth, $|\mathbf{s}_0\rangle$, defined as

$$\begin{aligned} |\mathbf{s}_0\rangle &= -xy\mathbf{i} + x\mathbf{j} - \mathbf{k} \\ |\mathbf{s}_1^{(1)}\rangle &= \mathbf{j} + x\mathbf{k} \\ |\mathbf{s}_2^{(1)}\rangle &= \mathbf{i} \\ |\mathbf{s}_3^{(1)}\rangle &= -x\mathbf{j} + \mathbf{k} \\ |\mathbf{s}_1^{(2)}\rangle &= y\mathbf{i} - \mathbf{j} \\ |\mathbf{s}_2^{(2)}\rangle &= \mathbf{i} + y\mathbf{j} \\ |\mathbf{s}_3^{(2)}\rangle &= \mathbf{k} \\ |\mathbf{s}_1^{(3)}\rangle &= -\mathbf{i} - y\mathbf{j} - xy\mathbf{k} \\ |\mathbf{s}_2^{(3)}\rangle &= -y(1+x^2)\mathbf{i} + (1-x^2y^2)\mathbf{j} + x(1+y^2)\mathbf{k} \\ |\mathbf{s}_3^{(3)}\rangle &= -xy^3(1+x^2)\mathbf{i} + x(1+2y^2+x^2y^2)\mathbf{j} - (1+y^2)\mathbf{k} \end{aligned} \quad (38)$$

for arbitrary numbers x & y . Now the angle ϕ between $|\mathbf{s}_0\rangle$ and $|\mathbf{s}_3^{(3)}\rangle$ found from their scalar product is given by relation

$$\cos(\phi) = [1 + x^2 + y^2 + x^2y^2(2 + x^2 + y^2 + x^2y^2)] / (||\mathbf{s}_0\rangle|| ||\mathbf{s}_3^{(3)}\rangle||) \quad (39)$$

whose rhs minimization is directly found at $\sqrt{8}/3$, obtained for $x = y = \pm 1$. Therefore, $0 \leq \phi \leq \cos^{-1}(\sqrt{8}/3)$. In addition to the explicit vector triplet constraints (23), by the same consequence, no perpendicular pair of vectors may be assigned number "1", since taken together with a mutually perpendicular third they would form a triplet. One finds then from the additional relations

$$\begin{aligned} |\mathbf{s}_0\rangle &\perp |\mathbf{s}_1^{(1)}\rangle, |\mathbf{s}_2^{(2)}\rangle, \text{ and } |\mathbf{s}_2^{(3)}\rangle \\ |\mathbf{s}_1^{(3)}\rangle &\perp |\mathbf{s}_3^{(1)}\rangle \text{ and } |\mathbf{s}_1^{(2)}\rangle \\ |\mathbf{s}_2^{(1)}\rangle &\perp |\mathbf{s}_3^{(2)}\rangle \end{aligned} \tag{40}$$

that the only possible assignments for the set of ten are those for which $|\mathbf{s}_0\rangle$ and $|\mathbf{s}_3^{(3)}\rangle$ have the same eigenvalue, $\nu(\mathbf{P}_0) = \nu(\mathbf{P}_3^{(3)}) = 0$ or 1 , demonstrated by simply trying out each of the other two possible assignments while respecting constraints implicit in (37) and (40). As only the relative $|\mathbf{s}_0\rangle$ to $|\mathbf{s}_3^{(3)}\rangle$ directions are relevant, the result establishes for KS that in realist thinking any two vectors in R3 of angular separation $0 \leq \phi \leq \cos^{-1}(\sqrt{8}/3)$ must have the same eigenvalue assignment, 0 or 1. We now take the full set $\{|\mathbf{s}_i^{(j)}\rangle\}$ of (38) for $\phi = 18^\circ (< \cos^{-1}(\sqrt{8}/3))$, and for convenience perform a rotation on the coordinate system such that $|\mathbf{s}_2^{(3)}\rangle$ points along the positive y-axis and $|\mathbf{s}_0\rangle$ along the positive z-axis. We show here three key vectors of the set scaled to 100, and table triplets according to context $\hat{\mathbf{B}}_i$.

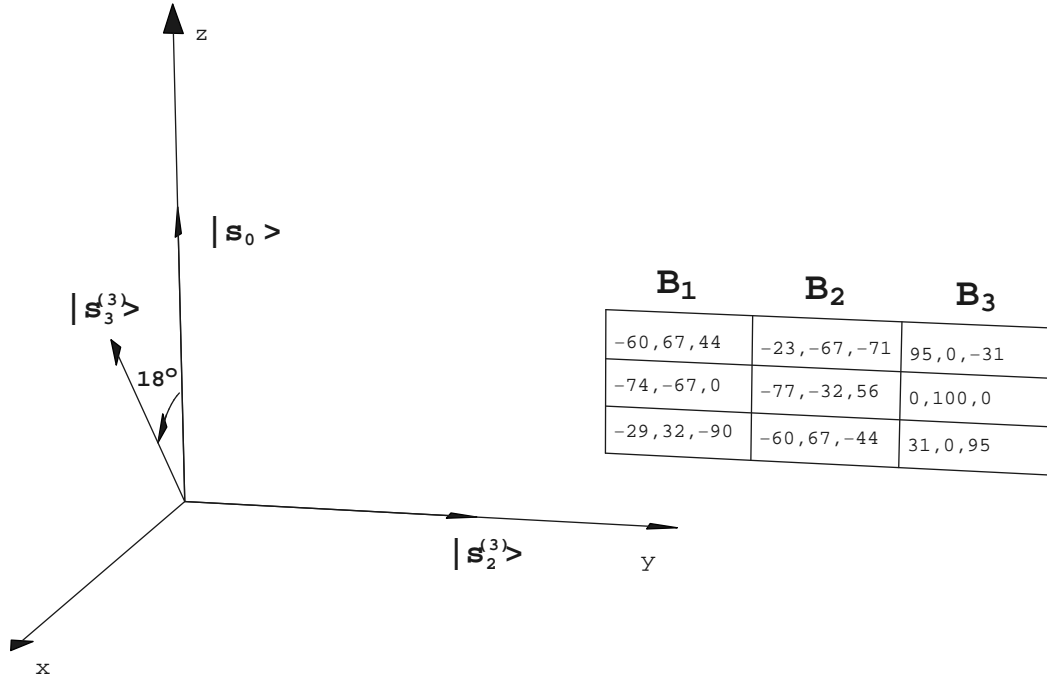


figure 11

From these we generate four new sets of nine vectors by rotating the original about $|\mathbf{s}_2^{(3)}>$ in four 18° increments, constraining successive $|\mathbf{s}_3^{(3)}>$'s thereby to the same value assignment, and generate five additional sets by first rotating the last set by 90° about its $|\mathbf{s}_3^{(15)}>$, then rotating in five more 18° increments about the rotated $|\mathbf{s}_2^{(15)}>$. For a final five we follow the same procedure to complete the first-quadrant sweep of $|\mathbf{s}_3>$: $|\mathbf{s}_3^{(15)}> (= |\mathbf{s}_2^{(33)}>) \rightarrow |\mathbf{s}_3^{(30)}> (= |\mathbf{s}_2^{(3)}>) \rightarrow |\mathbf{s}_3^{(45)}> (= |\mathbf{s}_2^{(18)}>)$, on the $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ triad

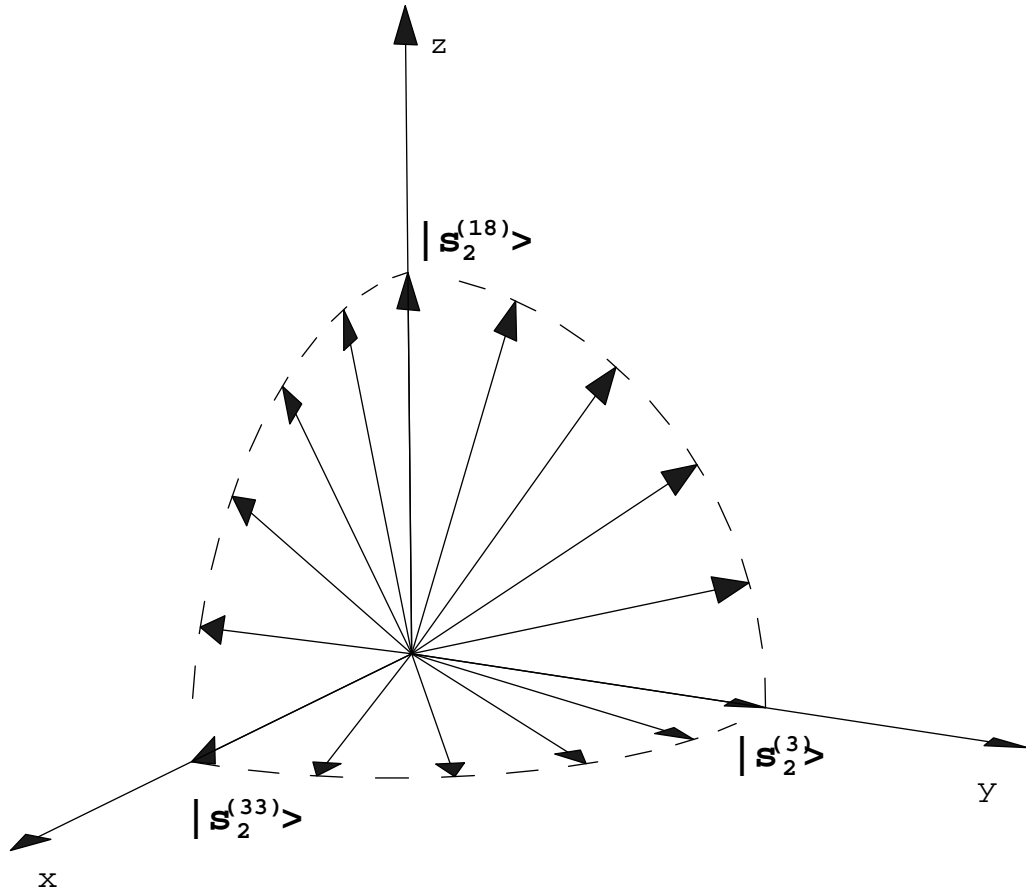


figure 12

each element of which, again, constrained to the same value assignment. But this immediately violates csr's (23) as $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ is itself a mutually perpendicular triplet. The violation is taken to demonstrate an inconsistency in the realist conception of reality as given by propositions p(1) and p(2) and thus concludes the KS paradox.

This result may also be illustrated visually. For the tabular form [14] consider first the vectors involved in the 15 orientations of the set of 9 as represented in figure (12); we list here the total 117 vectors (= $15 \times 9 - 15 - 3$, accounting also for the 15 $|s_2\rangle$ and 3 $|s_3\rangle$ overlaps), sub-grouping as in figure (11) according to context

B_1	B_2	B_3	B_{16}	B_{17}	B_{18}	B_{31}	B_{32}	B_{33}
(-60,67,44)	(-23,-67,-71)	(95,0,-31)	(44,-60,67)	(-71,-23,-67)	(-31,95,0)	(67,44,-60)	(-67,-71,-23)	(0,-31,95)
(-74,-67,0)	(-77,-32,56)	(0,100,0)	(0,-74,-67)	(56,-77,-32)	(0,0,100)	(-67,0,-74)	(-32,56,-77)	(100,0,0)
(-29,32,-90)	(-60,67,-44)	(31,0,95)	(-90,-29,32)	(-44,-60,67)	(95,31,0)	(32,-90,-29)	(67,-44,-60)	(0,95,31)

B_4	B_5	B_6	B_{19}	B_{20}	B_{21}	B_{34}	B_{35}	B_{36}
(-44,67,60)	(-44,-67,-60)	(81,0,-59)	(60,-44,67)	(-60,-44,-67)	(-59,81,0)	(67,60,-44)	(-67,-60,-44)	(0,-59,81)
(-71,-67,23)	(-56,-32,77)	(0,100,0)	(23,-71,-67)	(77,-56,-32)	(0,0,100)	(-67,23,-71)	(-32,77,-56)	(100,0,0)
(-56,32,-77)	(-71,67,-23)	(59,0,81)	(-77,-56,32)	(-23,-71,67)	(81,59,0)	(32,-77,-56)	(67,-23,-71)	(0,81,59)

B_7	B_8	B_9	B_{22}	B_{23}	B_{24}	B_{37}	B_{38}	B_{39}
(-23,67,71)	(-60,-67,-44)	(59,0,-81)	(71,-23,67)	(-44,-60,-67)	(-81,59,0)	(67,71,-23)	(-67,-44,-60)	(0,-81,59)
(-60,-67,44)	(-29,-32,90)	(0,100,0)	(44,-60,-67)	(90,-29,-32)	(0,0,100)	(-67,44,-60)	(-32,90,-29)	(100,0,0)
(-77,32,-56)	(-74,67,0)	(81,0,59)	(-56,-77,32)	(0,-74,67)	(59,81,0)	(32,-56,-77)	(67,0,-74)	(0,59,81)

B_{10}	B_{11}	B_{12}	B_{25}	B_{26}	B_{27}	B_{40}	B_{41}	B_{42}
(0,67,74)	(-71,-67,-23)	(31,0,-95)	(74,0,67)	(-23,-71,-67)	(-95,31,0)	(67,74,0)	(-67,-23,-71)	(0,-95,31)
(-44,-67,60)	(0,-32,95)	(0,100,0)	(60,-44,-67)	(95,0,-32)	(0,0,100)	(-67,60,-44)	(-32,95,0)	(100,0,0)
(-90,32,-29)	(-71,67,23)	(95,0,31)	(-29,-90,32)	(23,-71,67)	(31,95,0)	(32,-29,-90)	(67,23,-71)	(0,31,95)

B_{13}	B_{14}	B_{15}	B_{28}	B_{29}	B_{30}	B_{43}	B_{44}	B_{45}
(23,67,71)	(-74,-67,0)	(0,0,-100)	(71,23,67)	(0,-74,-67)	(-100,0,0)	(67,71,23)	(-67,0,-74)	(0,-100,0)
(-23,-67,71)	(29,-32,90)	(0,100,0)	(71,-23,-67)	(90,29,-32)	(0,0,100)	(-67,71,-23)	(-32,90,29)	(100,0,0)
(-95,32,0)	(-60,67,44)	(100,0,0)	(0,-95,32)	(44,-60,67)	(0,100,0)	(32,0,-95)	(67,44,-60)	(0,0,100)

table 4

Then from the following rules

1. The eigenvector whose projector has value 1(0) is assigned the color white (black).
2. Two orthogonal vectors may not both be colored white.
3. The assignments are independent of context.

we attempt to color the vector boxes of table 4. That the coloring assignments cannot be made consistently in accordance with the rules coincides with the paradox.

Finally, for a two-dimensional graphical form of the paradox [15] we represent vectors by small open circles, and orthogonality between pairs of vectors by a connecting straight line; a triangle with open circles at the vertices then represents a mutually orthogonal triplet, and our beginning set of vectors described in figure (11) has the KS diagram representation

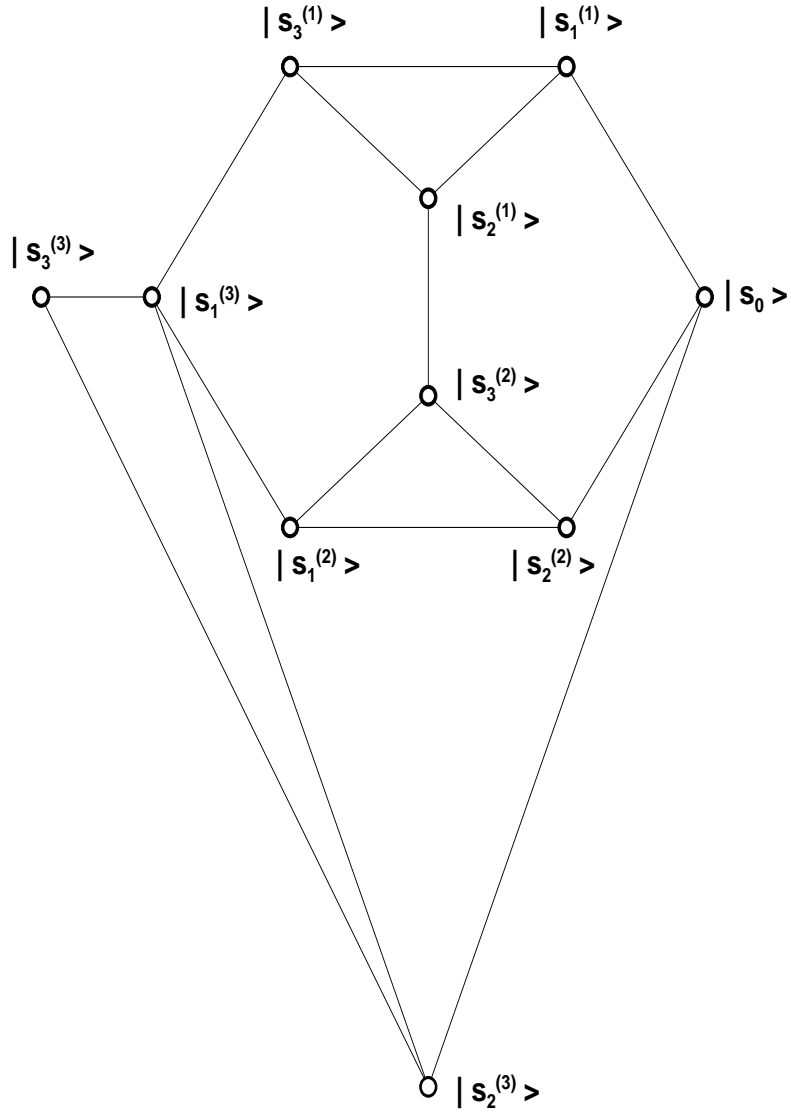


figure 13

Here the constraint that $|s_0>$ and $|s_3^{(3)}>$ have the same eigenvalue assignment for example follows from the inability to assign different colors to their circled vertices in accordance with the aforementioned coloring scheme for the corresponding vector boxes. If we then stretch the diagram a little to one side

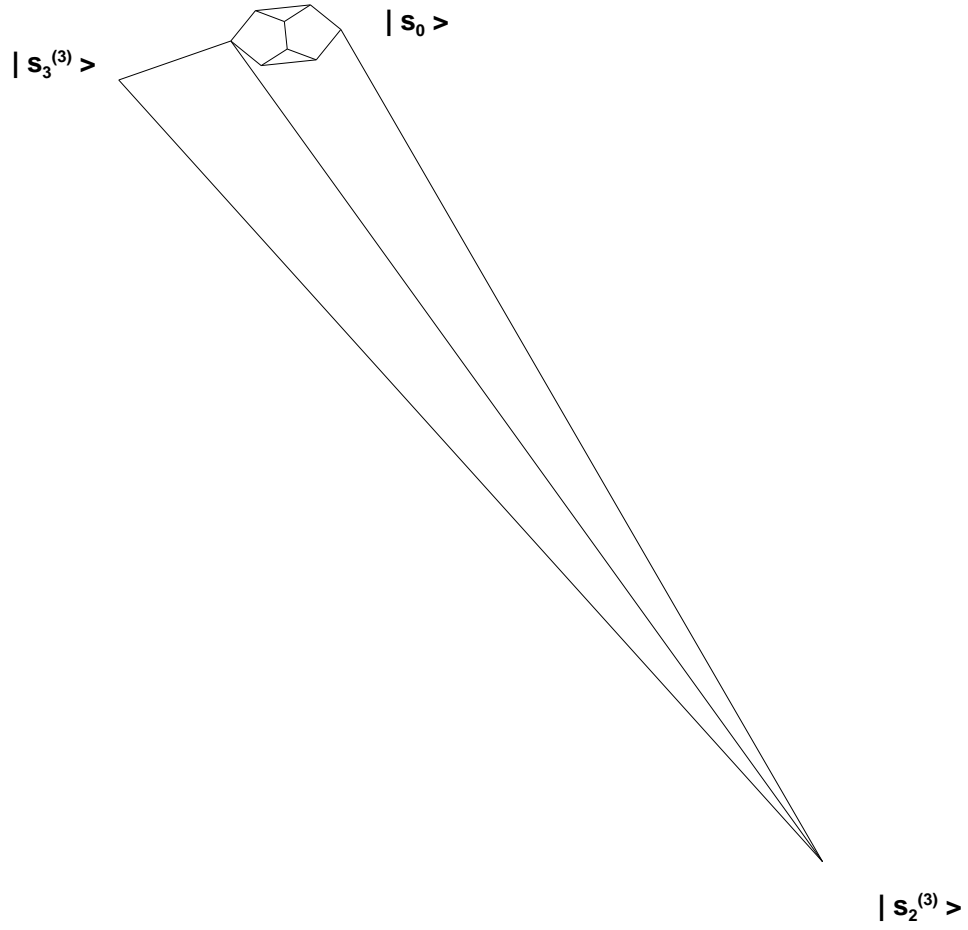


figure 14

and perform the 14 rotations about the appropriate $|s_2 >$ points (along with the two rotations about $|s_3 >$ points) as represented in figure 12, we generate the full KS diagram

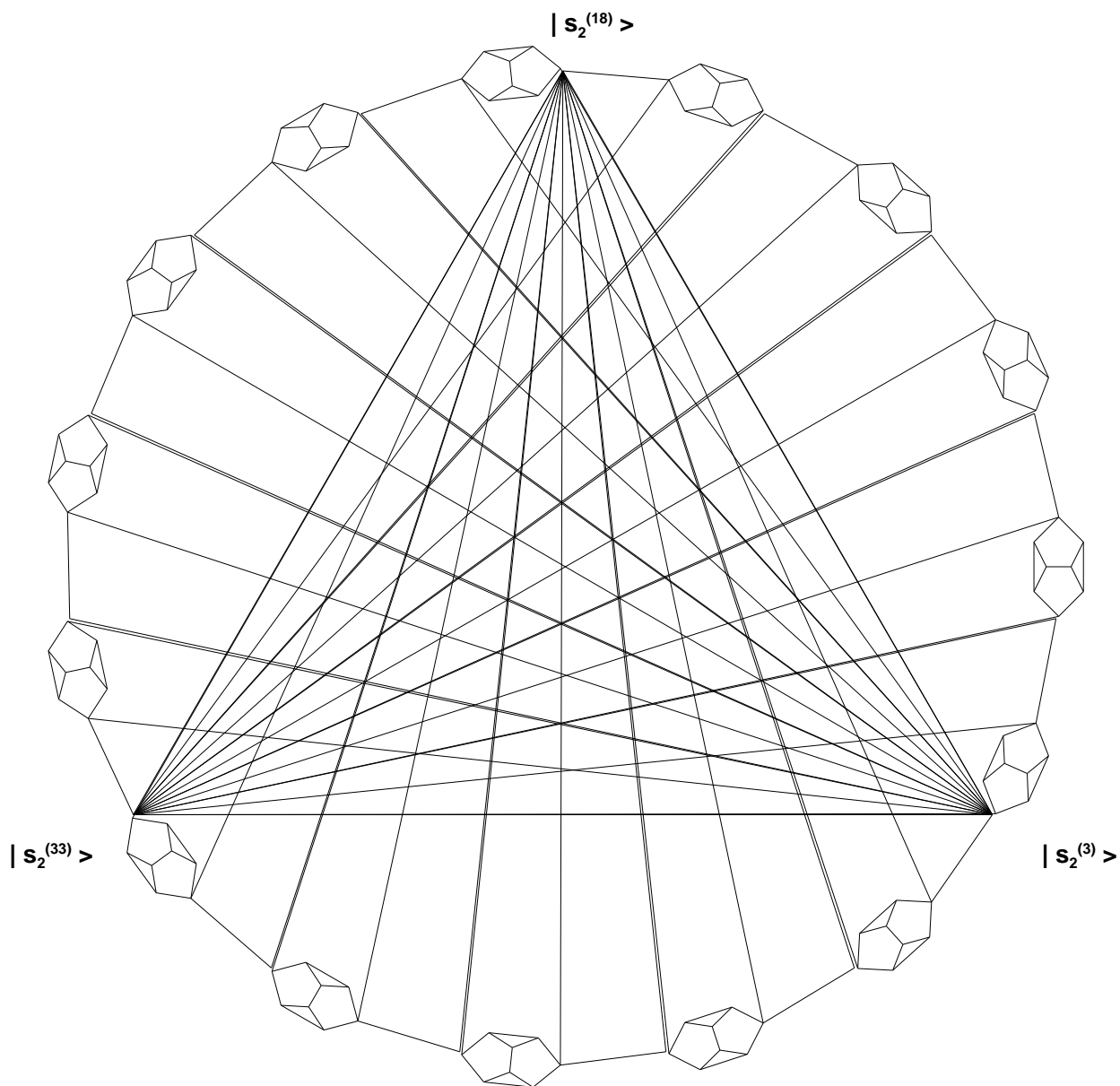


figure 15

said to resemble a cat's cradle. Here too, the inability to faithfully apply the coloring scheme to individual circled vertices is understood as a manifestation of inconsistency in tealist thinking on microscopic measurement. The difficulties presented by

these two visualizations are known as the KS *coloring problem*.

6.2 Von Neumann's paradox and resolution

In the Summer of 1932 eminent physicist and mathematician J. von Neumann devised a proof (VN) against the possibility of a consistent realist interpretation of QM [16], a precursor of the KS paradox, by way of the operator average relation

$$\mathbf{A} = \mathbf{B} + \mathbf{C} \Rightarrow \langle \Psi | \mathbf{A} | \Psi \rangle = \langle \Psi | \mathbf{B} | \Psi \rangle + \langle \Psi | \mathbf{C} | \Psi \rangle \quad (41)$$

similar to our equation (28) above. Operators \mathbf{B} and \mathbf{C} here do not commute, and the rhs terms are assumed to take on realist possessed values [7, p. 268]. This last condition is the VN additivity postulate P.IV from which derives the expectation value formula

$$\langle O \rangle = \text{Tr}(WO) \quad (42)$$

where Tr is the trace operator and W a statistical operator independent of observable O and descriptive of the system under observation. For the hypothetical value-definite states ψ , so-called "dispersion -free" states

$$\langle \psi | O^2 | \psi \rangle - \langle \psi | O | \psi \rangle^2 = 0 \quad (43)$$

in the language of the time, the formula proves problematic in the following way: The statistical operator describing the realist system ψ is just the projector $W_\psi = |\psi\rangle\langle\psi|$. Choosing for O then an arbitrary projector $|\phi\rangle\langle\phi|$ leads from (42) and (43) to the statement

$$\langle \phi | W_\psi | \phi \rangle = 0 \text{ or } 1 \quad (44)$$

for all possible QM states ϕ which cannot be true as the left-hand side is continuous in ϕ and the right hand side discontinuous. This casts doubts on the very existence of dispersion-free states W_ψ ⁷.

⁷I don't know that its been pointed out in the literature [17] that the above lhs continuity is consequent upon a separate application of additivity by which two dispersion-free states of opposite

This VN impossibility proof has long been understood to follow from a misapplication of its additivity postulate ⁸ which is experimentally valid for expectation values, though, as shown in section 5.2, not applicable to real-possessed values. The violation follows from simple statistics: ensemble values ($\sim \Psi^2$) are generally not identical with ensemble-member values ($\sim \psi^2$). The Von Neumann paradox rests upon an illicit application of proposition p(1) to quantum mechanical sub-ensembles via equations of QM which Ψ alone satisfies:

$$\Psi + p(1) \Rightarrow \text{VN Paradox}$$

(Ψ implying p(2)). It has the schematic resolution

$$\Psi \rightarrow \psi$$

or equally well,

$$p(1) \rightarrow p(2).$$

We have then

$$\left. \begin{array}{c} \psi + p(1) \\ \text{or} \\ \Psi + p(2) \end{array} \right\} \Rightarrow \text{no VN Paradox} \quad (45)$$

6.3 Gleason's result

In response to the objectionable additivity postulate necessary to the VN proof, A. M. Gleason derived the main there formula (42) by means of an acceptable additivity for commuting operators [18], as appears e.g. in the above csr's (23), in eigenvalues, ψ_1 and ψ_0 , are joined by a connecting vector $\Psi(x) = a_x\psi_0 + b_x\psi_1$, for properly chosen functions a_x and b_x (See e.g. reference [7, p. 294]). Such an operator, $O = |\Psi\rangle\langle\Psi|$ is not an observable in the Hilbert space of dispersion-free states under consideration. In light of this, the von Neumann proof might be understood to follow from an assumption, under additivity, of the existence of dispersive states in, under realism, the Hilbert space of dispersive-free states: $\langle\psi|\Psi\rangle = 0$ or 1 for all Ψ .

⁸See J. Bell, ref.[11]. A more thorough exposition is given in [6, pp. 25-34]

addition to a tacit assumption of noncontextuality first noticed only years later by Bell [11]. By demonstrating via (42) the necessary continuity of mappings from H_3 and higher dimensions to R_1 , i.e. that Hilbert space projection operators P_i are mapped continuously onto the closed interval $[1,0]$, Gleason proves under noncontextuality the impossibility of the realist necessarily discontinuous mapping $\nu : P \rightarrow R_1$, $\nu(P) = 0$ or 1 , and so the non-existence of ψ and an inconsistency in realist hidden variable theories. The key result of Kochen and Specker, constraint (36), then gives an upper-bound estimate on the $\nu'(\phi)$ implicit to noncontextual value-definite hidden variable theories.

7 the KS paradox reconsidered

The KS proof improves on the VN proof in that it in effect applies the constraints of value definiteness, proposition p(1), appropriately to the hypothetical realist state function ψ ($\sim \nu$). Proposition p(2), noncontextuality, on the other hand has largely been inferred indirectly as an extension to realist thinking. For example, from Einstein's rhetorical

Do you believe the moon is really there if no one is looking? [19] (e)

which suggests an independence of reality from its observation.

Let us briefly take a closer look. As the statement addresses the question of existence itself, the observation-independence or noncontextuality inferred from it is ontological⁹: There is an objective reality "out there" with an *existence* independent of observation¹⁰. The aforementioned reality criterion of section 5.1, however concerns an observation dependence of a different, more mundane sort: The observation-dependence of states of existence, of *states* (which addresses e.g. the

⁹Ontological in the sense that it concerns the principles and causes of being.... Unfortunately for the writer, the term "ontological contextualism" is already in use [20] though with a meaning it seems closer to that of causal contextuality. I use it here sparingly in the first sense.

¹⁰See e.g. K. Popper in ref.[21, p. 2]

question whether an electron's spin is affected by an observation of it), not of the very existence of the thing observed. The clear indication is that, notwithstanding the select elements that meet the reality criterion - a sufficient criterion - the value of a physical quantity does indeed depend upon how it is obtained, upon the manner in which the object under observation is affected, disturbed during measurement. Such a contextual dependence is causal and the values thus measured causally contextual - precisely the sort of effect that experimentalists in EPR tests go to great lengths to guard against [22]. As J. Bell writes, "The result of an observation may reasonably depend not only on the state of the system... but also on the complete disposition of the [measuring] apparatus." [11]. But these categories are confused in the KS and Gleason analysis, realist dismissal of ontological contextuality misinterpreted as a dismissal of causal contextuality, leading, finally, to mistaking $p(2)$ as a realist proposition.

While the von Neumann proof relies on an explicit misapplication of realist value-definiteness within the QM formalism, as shown in 5.1, in its misapplication of additivity via the equations of QM such as (27) to realist states ψ , it is also seen to violate causal contextuality; in the single QM expectation-value equation (29) the relevant contextual effects are washed out in the average. KS and Gleason achieve the same ends by means of properly single context equations, though several taken simultaneously in terms of the csr's of (23) for mutually exclusive contexts. And so in principle both commit the same violation in unduly constraining the most general causality. Again, to illustrate, there is nothing in realist thinking on the spin-1 measurements

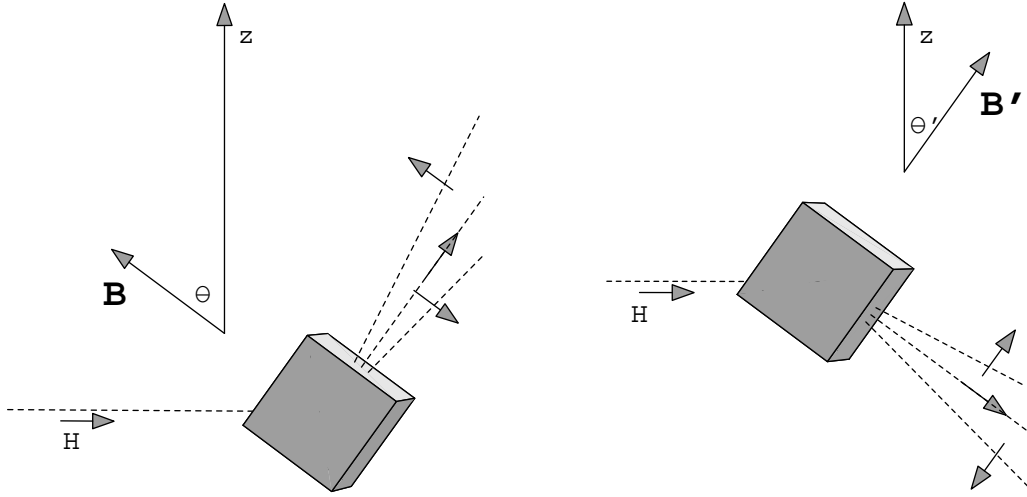


figure 16

($\theta - \theta' = 90^\circ$) to suggest the counterfactual implied in the noncontextual constraint $\nu(P_1) = \nu'(P_2)$ (or, with reference to figure 10, the constraint $\nu(P_2) = \nu'(P_2)$).

The KS diagram of figure (15) describes a set of experiments each of which may in principle be performed; accordingly, the diagram itself may always be constructed. Its coloring however depends somewhat upon interpretation of QM wavefunction. While both orthodox and realist interpretations achieve coloring consistency by means of various shadings, indeed in violation of the coloring rules, their mechanisms differ: Indeterminism on one hand, causal contextuality on the other. At issue here is the KS contention that while orthodox coloring is constrained only by the rules governing individual triplets, those that might be derived from the csr (18), realist coloring is constrained by the full KS set.

Proposition p(2) from which in combination with p(1) the rules follow does not properly apply to individual ensemble-member values, realist possessed values ν , and it is precisely this misapplication that is at the heart of the KS paradox.

The paradox may thus be understood from the following schematic:

$$\psi + p(2) \Rightarrow \text{KS Paradox}$$

(ψ implying $p(1)$). It is resolved by the replacement

$$\psi \rightarrow \Psi$$

Taken together then

$$\left. \begin{array}{c} \Psi + p(2) \\ \text{or} \\ \psi + p(1) \end{array} \right\} \Rightarrow \text{no KS Paradox} \quad (46)$$

8 conclusion

While quantum mechanics is generally accepted as the best tool available for the prediction of small-scale statistics, there remains disagreement over its physical interpretation particularly as regards its Hilbert-space state vectors. In the prevailing orthodox view Ψ is said to represent individual systems e.g. an individual electron of an ensemble. It is understood in consequence that individual systems cannot consistently possess many of the common characteristics we attribute to them in ordinary language use (spatial position, mass, spin, etc.). The conceptual challenges presented by the view are obvious and well-known and an entire culture in science thinking has gone to their explication [21, 7]. While this particular concept does indeed appear common to all orthodox-school thinking we caution not to overgeneralize with regard what is really a diversity of views [23]. Likewise in characterizing the realist view ¹¹, particularly as impossibility proofs such as the ones here reviewed are directed against their very logic. Let us therefore carefully consider again the type of experimental arrangement cited earlier in illustration of realist thinking on the fundamental issue here, noncontextuality.

For the pair of illustrated spin-1 projection measurements

¹¹See refs.[15, p. 44] and [6, p. 9]

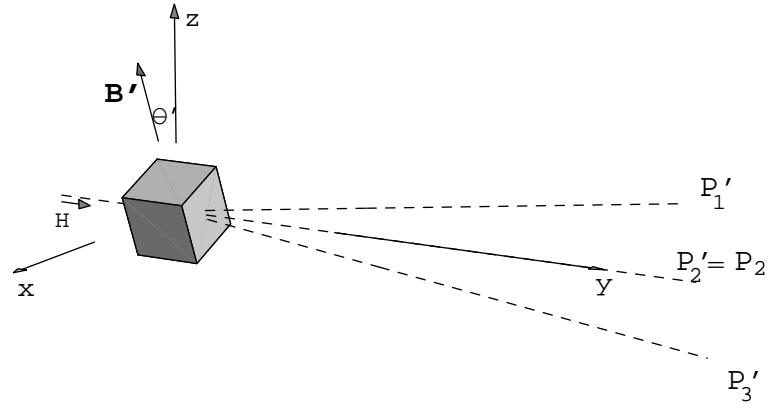
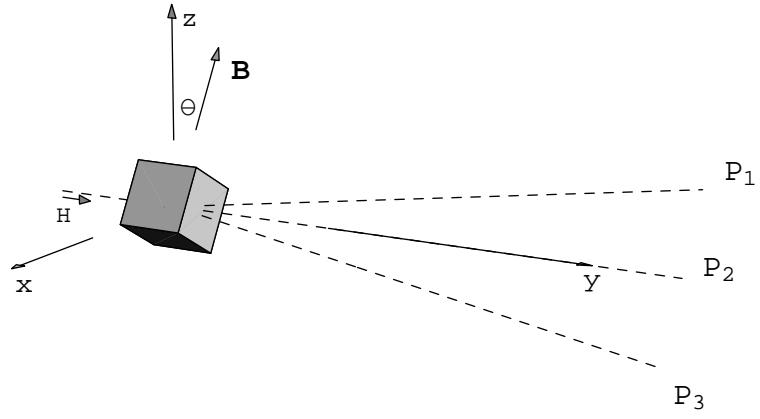


figure 17

one always has the possessed-value relations

$$\nu(P_1) + \nu(P_2) + \nu(P_3) = 1 \quad (47)$$

and

$$\nu'(P'_1) + \nu'(P'_2) + \nu'(P'_3) = 1 \quad (48)$$

We ask now whether in the realist view a result of measuring the projection $P_2 = P'_2$ may depend upon how the measurement is taken. Whether

$$\nu(P_2) = \nu'(P'_2) \quad (49)$$

by necessity. Based on statements of the generality (e) above KS assumes that the realist view answers in the affirmative, the claim (not to beat a dead horse) at the heart of the derivation of their paradox, as shown in section 6.1.

But the question is not unique to experimental arrangements of the type pictured in figure (17). The very same also appears in Bell -Inequality (BI) analysis [11] in the context of the question of space-time locality ¹². There, a set of fixed projection measurements is taken (by detector D_1 at space-time position x) on spin-1/2 particles, as the setting of a remote apparatus (D_2 at y) is varied

¹²There, it is certainly true that what is to the realist view a question of contextuality (of the possible effects of D_2, ψ_2 , etc. on ψ_1 , *assuming separability* - that the effects between paired particles are purely causal) is to the orthodox view one of separability only (roughly, of the affects of decay particle ψ_2 on particle ψ_1 intrinsic to their bound state). Hence e.g. the realist issue with detector D_2 triggering.

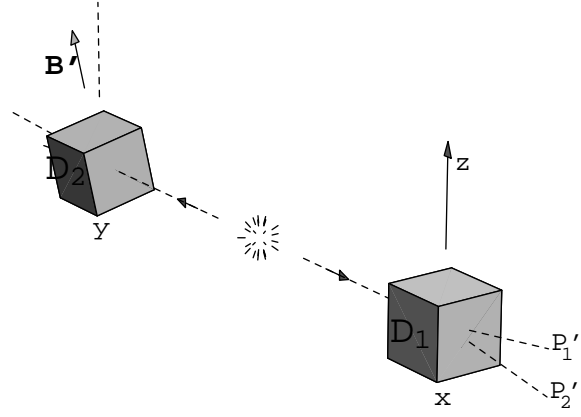
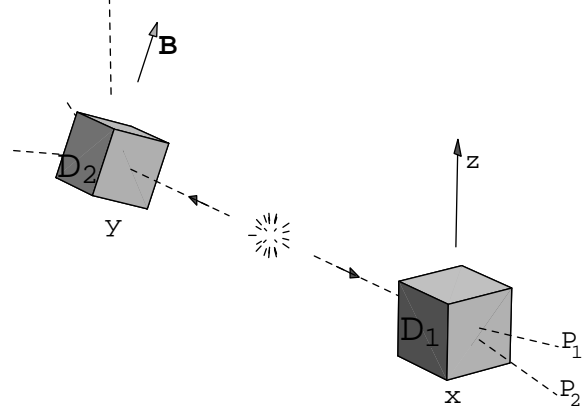


figure 18

One then considers contextuality question, whether

$$\nu_{\mathbf{k}}(P_1) = \nu'_{\mathbf{k}}(P_1) \quad (50)$$

where here too value subscripts and superscripts in obvious notation characterize

the measurement context ¹³.

An affirmative answer without condition leads as is well-known to an realization of the Bell inequality and immediate disagreement of realism with observation. And no one has thought to do this. On the contrary, contextuality is assumed as a matter of course, its possible effect dismissed only upon satisfaction of the most stringent locality conditions

$$(x - y)^2 > 1 \quad (51)$$

brought to some level of experimental realization in the work of Aspect et al and several refinements since [22]. We are keen to make this point, as (to continue the beating) causal contextuality is indeed central to realist thinking particularly since the appearance of quantum phenomena and the realist conceptual involvement of hidden variables. D. Bohm confirms that "when we measure the momentum 'observable,' the final result is determined by hidden parameters in the momentum-measuring device as well as by hidden parameters in the observed electron" [24]. N. Bohr also, himself no realist, agrees ¹⁴, while others, more often than not on purely formalistic grounds, may not [26]. We conclude therefore that the value-definiteness of realism p(1) is in no way incompatible with causal contextuality not-p(2) but is on the contrary complemented by it ¹⁵.

If figure(18) by comparison with (17) (in light of constraint (51)) is taken to indicate an experimental constraint appropriate to noncontextuality in the KS analysis, the

¹³While it is possible for individual states and measurements (ψ , ν) to depend upon context while those of ensembles (Ψ , $\langle \rangle$) remain independent, or noncontextual, Bell's inequality is concerned with ensemble contextuality (that of Ψ) though by way of individual state contextuality (that of ψ). The KS paradox on the other hand treats the contextuality of individual states.

¹⁴"This crucial point... implies the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.", Ref. [25, p. 209]

¹⁵Without naming it, as the term had yet to enter the language, Belinfante offers in part 1 of reference [6] (up to page 78) an excellent insight into contextual thinking. Likewise Bell [11], the first to notice contextuality as an issue central to the KS paradox. There are many others.

feasibility of such a test seems clearly in doubt. We will not pursue the question here.

What is striking about the realist view of possessed values is its marked lack of novelty. The possessed values of an electron are thought to be no different in their manner of possession than those of tables, trees, cats; a stone said to possess 10 kilograms is said in the same breath to weigh 98 Newtons, to oscillate with a frequency $f = \frac{1}{2\pi} \sqrt{\frac{k}{10}}$ when attached to a spring k , and so forth; i.e., such a stone possesses all the values that a stone of 10 kilograms might yield upon measurement, many of which from the laws of mechanics may be known before hand.... To say that the stone is 10 kilograms is a manner of speaking, a shorthand for an otherwise unwieldy usage.

In absence such a shorthand however one is left with the directly observed or predicted values themselves, no less possessed by the object. In this form they may well appear in some ways peculiar and to have unusual properties¹⁶. They do not; and the appearance generally dissolves upon careful consideration: Again, the manner of microscopic property possession in this view is no different than that for macroscopic possession.

It has long been considered not possible to assign definite values to systems whose possessed values are described by noncommuting QM operators. The KS paradox proves this for a discrete set of observables and as such is a statement of the limitation of QM to predict individual microscopic phenomena. But the paradox is not a proper criticism of realist thinking; it reduces to absurdity a view held by no one.

¹⁶van Fraassen's ontological contextualism has taken special ridicule for its proliferation of possessed values that e.g. "pop in and out of existence" at the whim of the experimentalist [14], "de-Ockhamizing QM!" [15, p. 135]. Such a criticism on mathematical form, Bell might have countered, merely betrays a lack of sufficient imagination [28, p. 64]. Einstein too was largely unconcerned with formal structures per se [7, pp. 234-238].

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